

Generalized quantal distribution functions within factorization approach: Some general results for bosons and fermions

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The generalized quantal distribution functions are investigated concerning systems of non-interacting bosons and fermions. The formulae for the number of particles and energy are presented and applications to the Chandrasekhar limit of white dwarfs stars and to the Bose-Einstein condensation are commented.

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I. INTRODUCTION

Since 1988, a non-extensive formalism of statistical mechanics [1] has been developed and up to recent years, many works have been devoted to show the robustness and usefulness of this approach. We believe it is robust in the sense it allows generalizations of a variety of fundamental concepts of statistical thermodynamics [2], such that it avoid to enter in severe contradiction with well established facts. In addition, we believe it is useful in the sense it provides a theoretical basis and relevant explanation of some experimental and observational situations [3], where Boltzmann-Gibbs statistics fails. That is to say, where Boltzmann-Gibbs statistical functions diverge and do not yield any physical prediction.

However, this formalism, unlike Boltzmann-Gibbs', has a non-zero set of free parameters, here represented by q ($q \in \mathbb{R}$). This unique parameter controls the degree of the nonextensivity of the system in consideration. (At this point, it is worth noting that the formalism includes the standard, extensive, statistics as a special case for the value of $q = 1$ and all expressions derived within this non-extensive framework give the results of Boltzmann-Gibbs statistics in the $q \rightarrow 1$ limit.) Only after 1995, some works started to address the long standing puzzle of understanding the physical meaning of q . Amongst the works related to this topic, two main streams started to become more apparent. On one side, there are attempts on the study of conservative [4] and dynamical systems, more precisely, dissipative systems with both, low [5] and high [6] dimensions. On the other, there have been some efforts of estimating bounds upon q in measurable physical systems. The physical applications studied so far can be enumerated as follows: the microwave background radiation [7,8], the Stefan-Boltzmann constant [9,10], the early Universe [11,12] and the primordial neutron to baryon ratio in a cosmological expanding background [13]. As a final note, a recent letter by Alemany [14] explored the definition of a new *fractal canonical ensemble*, associated with the parameter q . However, first estimations for the universe as a whole yielded to values of q bigger than those allowed by nucleosynthesis [13].

In all these works, two different approaches for the quantal distribution functions have been used: a closed analytical form for them is still lacking. The asymptotic approach of the kind $\beta(1 - q) \rightarrow 0$ of Tsallis et al. [7] was used in Refs. [7,9,11,13] and also in some other applications such as the study of Bose-Einstein condensation in a fractal space [15], the specific heat of ^4He [16] and the thermalization of an electron-phonon system [17]. The second approach for the generalized distribution functions has been proposed by Büyükkılıç et al. [18], which we refer to as *Factorization Approach*. This last term is justified in that it is generically based on the factorization of the generalized grand canonical distribution and the concomitant generalized partition function as if they were extensive quantities. However, although the generalized distribution functions within factorization approach have been derived before the asymptotic approach, they have not been preferred to be used in physical applications, mainly due to a work by Pennini et al. [19]. There, the authors claimed that the quality of the results of the factorization approach deteriorates as the number of the particles of the system increases. But very recently, contrary to the previous belief, new results by Wang and Lé Méhauté [20], favoured the factorization approach and showed clearly that there exists a temperature interval where the ignorance of the approximation is significant, but otherwise, the results of this approach can be used with confidence no matter the number of particles. In fact, they showed that, for a macroscopic system (i.e., with $\sim 10^{23}$ particles) having two states with a small energy interval of about 10^{10} Hz, this forbidden temperature zone is very narrow and situated at extremely high temperatures ($\sim 10^{20}$ K) and as the number of levels increases the forbidden temperature zone shifts to higher temperatures. Therefore the results of the factorization approach can be used at low temperatures up to 10^{20} K without any ignorance for any physical system under consideration [20–22]. As it was expected, the study of Wang and Lé Méhauté started to accelerate new attempts of applying these distribution functions to physical systems in order to estimate some alternative bounds upon q [10,12]; being this approach more handable than the one advanced by Tsallis et al. Although, after Wang and Lé Méhauté [20], some physical systems have been worked out with the help of the generalized distribution functions of the factorization approach, general expressions for magnitudes of fermions and bosons are still lacking.

Indeed, the possible need for a nonextensive formalism of thermostatistics is clear for a long time in gravitation [23], magnetic systems [24], Lévy-like diffusions [25], some surface-tension problems [26], etc., since Boltzmann-Gibbs formalism is known to fail whenever the physical system under consideration includes (i) long-range interactions and/or (ii) long-memory effects and/or (iii) the system evolves in a multifractal-like space-time. The last property can be understood as the system evolves in a porous-like medium where the properties of the multifractal structures govern it.

In the light of the facts stated above, we address the study of the generalized quantal distribution functions of fermions and bosons in a non-Euclidean, fractal-like space-time and to obtain general results for magnitudes, such as the number of particles and the internal energy that could be used in general for

any application to generalized systems. Once these general formulae be established, we proceed further investigating how some simple constructs of the statistical theory are affected by the change of distribution functions, for instance, the Chandrasekhar mass and the Bose-Einstein condensation. A similar work studying the Bose-Einstein condensation was recently done, as commented above, by Curilef [15] and our results are compatible in this case. Finally, it is worth noting that, for non-interacting fermions and bosons, the information about the fractal structure is kept in the nonextensivity parameter q , which is by now verified to be connected with the fractal dimension and the multifractal singularity spectrum [5]. In a sense, we shall study if any correction to the usual number of particles and energy arises due to the change of the distribution functions.

II. THE FERMION CASE

Let us start by writing down the generalized distribution function for fermions, up to $(1 - q)$ order. It is given by [18],

$$n_{q[fermions]} = \frac{1}{e^{\beta(\epsilon - \mu)} + 1} + \frac{q - 1}{2} \frac{[\beta(\epsilon - \mu)]^2 e^{\beta(\epsilon - \mu)}}{(e^{\beta(\epsilon - \mu)} + 1)^2}, \quad (1)$$

where $\beta = 1/kT$, ϵ is the energy level and μ is the chemical potential.

We are interested in solving the following integrals. For the number of particles,

$$\langle N \rangle = \frac{4\pi V(2m)^{3/2}}{h^3} \left[\int_0^\infty d\epsilon \frac{\epsilon^{1/2}}{e^{\beta(\epsilon - \mu)} + 1} + \frac{q - 1}{2} \int_0^\infty d\epsilon \frac{(\beta(\epsilon - \mu))^2 e^{\beta(\epsilon - \mu)} \epsilon^{1/2}}{(e^{\beta(\epsilon - \mu)} + 1)^2} \right] \quad (2)$$

and a similar one, replacing the factors $\epsilon^{1/2}$ for $\epsilon^{3/2}$, for the energy $\langle U \rangle$. Note that we have assumed the same pre-factor for both terms. This can be justified if one thinks of this factor as coming from microscopic quantum considerations and a change of variables, from momentum to energy, inside the integrals. See for instance, section 2.4 of Ref. [27].

Then, we can split $\langle N \rangle$ into two parts. Firstly we have the usual expression, given by the first term in the rhs of (2),

$$Usual = \frac{4\pi V(2m)^{3/2}}{h^3} \int_0^\infty d\epsilon \frac{\epsilon^{1/2}}{e^{\beta(\epsilon - \mu)} + 1} = \frac{8\pi V(2m\mu)^{3/2}}{3h^3} \left[1 + \frac{1}{8} \left(\frac{\pi^2}{\beta\mu} \right)^2 + \frac{7}{640} \left(\frac{\pi}{\beta\mu} \right)^4 + \dots \right]. \quad (3)$$

And we also have a second term,

$$I_q = \frac{4\pi V(2m)^{3/2}}{h^3} \frac{q - 1}{2} \int_0^\infty d\epsilon \frac{(\beta(\epsilon - \mu))^2 e^{\beta(\epsilon - \mu)} \epsilon^{1/2}}{(e^{\beta(\epsilon - \mu)} + 1)^2} = \frac{4\pi V(2m)^{3/2}}{h^3} \frac{q - 1}{2} I. \quad (4)$$

I can be recasted using the following dimensionless variables:

$$x = \beta\epsilon, \quad \psi = \beta\mu. \quad (5)$$

This enables us to write

$$I = \beta^{-3/2} \int_0^\infty \frac{(x - \psi)^2 e^{x - \psi} x^{1/2}}{(e^{x - \psi} + 1)^2} dx = \beta^{-3/2} I_x. \quad (6)$$

In order to solve this last I_x we apply the following trick. We know that integrals of the form

$$\int_0^\infty \frac{f(x)}{e^{x - \psi} + 1} dx \quad (7)$$

have a solution,

$$\int_0^\psi f(x)dx + \frac{\pi^2}{6} \frac{df}{dx} + \frac{7\pi^4}{360} \frac{d^3f}{dx^3} + \frac{31\pi^6}{15120} \frac{d^5f}{dx^5} + \dots, \quad (8)$$

where the derivatives have to be evaluated in $x = \psi$ [27]. In fact, this was already used to write down the solution for the usual term of (2). The idea is then to introduce an extra parameter, l , in the integral (7) and consider,

$$\int_0^\infty \frac{f(x)}{e^{x-l\psi} + 1} dx. \quad (9)$$

Here, it is important to stress that $f(x)$ does not depend on l . If we derive inside this integral with respect to l , the result will be

$$\int_0^\infty \frac{f(x)\psi e^{x-l\psi}}{(e^{x-l\psi} + 1)^2} dx, \quad (10)$$

and if $l = 1$, it can reproduce our integral I_x iff the function $f(x)$ is given by,

$$f(x) = \frac{(x - \psi)^2}{\psi} x^{1/2}. \quad (11)$$

Thus, using (11), the solution to I_x is,

$$I_x = \left[\frac{d}{dl} \int_0^\infty \frac{f(x)}{(e^{x-l\psi} + 1)} dx \right]_{l=1}. \quad (12)$$

We have to be very careful in using the correct formula to get the integral in $f(x)$: we should redefine $\tilde{\psi} = l\psi$ and apply then (8). So, the mechanism is, considering $f(x)$ as in (11), compute the integral (9), derive the result with respect to l and finally evaluate in $l = 1$. That yields, as we have seen, the integral we need.¹

Let treat each of the term in (8) separately. The first term is obtained immediately:

$$\int_0^{\tilde{\psi}} \frac{(x - \psi)^2}{\psi} x^{1/2} dx = \frac{1}{\psi} \int_0^{\tilde{\psi}} (x^2 - 2x\psi + \psi^2) x^{1/2} dx, \quad (14)$$

which finally leads to,

$$1^{st} \text{ term} = \frac{1}{\psi} \left[\frac{\tilde{\psi}^{7/2}}{7/2} - 2\psi \frac{\tilde{\psi}^{5/2}}{5/2} + \psi^2 \frac{\tilde{\psi}^{3/2}}{3/2} \right]. \quad (15)$$

Now we have to derive with respect to l (recall that $\tilde{\psi} = l\psi$), obtaining

$$\frac{d \text{ 1}^{st} \text{ term}}{dl} = \frac{1}{\psi} \left[(7/2)l^{5/2} \frac{\psi^{7/2}}{(7/2)} - 2(5/2)l^{3/2} \frac{\psi^{7/2}}{(5/2)} + (3/2)l^{1/2} \frac{\psi^{7/2}}{(3/2)} \right], \quad (16)$$

and evaluate at $l = 1$. The result is,

$$\left[\frac{d \text{ 1}^{st} \text{ term}}{dl} \right]_{l=1} = 0. \quad (17)$$

¹A more transparent way of solving these integrals (see acknowledgments) is to take the integral (6) as

$$\frac{\partial}{\partial \psi} \int_0^\infty \frac{x^{5/2}}{e^{x-\psi} + 1} dx - 2\psi \frac{\partial}{\partial \psi} \int_0^\infty \frac{x^{3/2}}{e^{x-\psi} + 1} dx + \psi^2 \frac{\partial}{\partial \psi} \int_0^\infty \frac{x^{1/2}}{e^{x-\psi} + 1} dx \quad (13)$$

and apply expansion (8) to each part. As we see, in fact there is no need to introduce such an extra parameter l , and it only stands as a mathematical trick in a particular form of solving the integrals. Numerical results for the coefficients obtained with both methods are the same. We warn the reader not to be distracted for such a trick.

The second term in (8) is obtained as,

$$2^{nd} \text{ term} = \frac{\pi^2}{6} \left[2 \frac{(x-\psi)}{\psi} x^{1/2} + \frac{(x-\psi)^2}{\psi} \frac{1}{2} x^{-1/2} \right]_{\tilde{\psi}}, \quad (18)$$

which finally yields to,

$$2^{nd} \text{ term} = \frac{\pi^2}{6} \left[2 \frac{(l\psi-\psi)}{\psi} (l\psi)^{1/2} + \frac{(l\psi-\psi)^2}{\psi} \frac{1}{2} (l\psi)^{-1/2} \right]. \quad (19)$$

The derivative with respect to l is,

$$\begin{aligned} \frac{6}{\pi^2} \frac{d \text{ 2}^{nd} \text{ term}}{dl} &= 2(l\psi)^{1/2} + 2(l\psi-\psi) \frac{1}{2} (l\psi)^{-1/2} + 2(l\psi-\psi) \frac{1}{2} (l\psi)^{-1/2} \\ &\quad + (l\psi-\psi)^2 \frac{1}{2} l^{-3/2} \frac{-1}{2} \psi^{-3/2}. \end{aligned} \quad (20)$$

Evaluating in $l = 1$ we obtain,

$$\left[\frac{d \text{ 2}^{nd} \text{ term}}{dl} \right]_{l=1} = \frac{\pi^2}{3} \psi^{1/2}. \quad (21)$$

One can do the same with the third term. In this case we need to compute the third derivate. It is given by,

$$f^{iii} = \frac{3}{\psi} x^{-1/2} - \frac{3}{2} (x-\psi) \frac{x^{-3/2}}{\psi} + (x-\psi)^2 \frac{3}{8} \frac{x^{-5/2}}{\psi}. \quad (22)$$

Computing this last for $x = \tilde{\psi}$ and deriving with respect to l we get,

$$\frac{df^{iii}}{dl} = -3(l\psi)^{-3/2} + 3(l\psi-\psi)(l\psi)^{-5/2} + (l\psi-\psi)^2 \left(\frac{-15}{16} \right) (l\psi)^{-7/2}. \quad (23)$$

And finally making $l = 1$, the correction is found to be,

$$\left[\frac{df^{iii}}{dl} \right]_{l=1} = -3(\psi)^{-3/2}. \quad (24)$$

Now we can go back and collect the results for I , which results,

$$I = \beta^{-3/2} I_x = \beta^{-3/2} \left(0 + \frac{\pi^2}{3} \psi^{1/2} - \frac{7\pi^4}{120} \psi^{-3/2} + \dots \right), \quad (25)$$

and the final expression for $\langle N \rangle$ is then,

$$\begin{aligned} \langle N \rangle &= \frac{3h^3}{8\pi V(2m\mu)^{3/2}} = \left(1 + \frac{1}{8} \left(\frac{\pi^2}{\beta\mu} \right)^2 + \frac{7}{640} \left(\frac{\pi}{\beta\mu} \right)^4 + \dots \right) \\ &\quad + \frac{q-1}{2} \left(0 + \frac{\pi^2}{2} \frac{1}{\beta\mu} - \frac{21\pi^4}{240} \frac{1}{(\beta\mu)^3} + \dots \right). \end{aligned} \quad (26)$$

This final expression represents the correction terms due to non-extensivity that arise for $\langle N \rangle$.

We can do exactly the same for the energy $\langle U \rangle$. I_q is given now by,

$$I_q = \frac{4\pi V(2m)^{3/2}}{h^3} \frac{q-1}{2} \beta^{-5/2} I_x, \quad (27)$$

where I_x is,

$$I_x = \int_0^\infty \frac{(x-\psi)^2 e^{x-\psi} x^{3/2}}{(e^{x-\psi} + 1)^2} dx. \quad (28)$$

We apply again the same trick, in this case, with a function $g(x) = (1/\psi)(x - \psi)^2 x^{3/2}$. As before,

$$I_x = \left[\frac{d}{dl} \int_0^\infty \frac{g(x)}{(e^{x-l\psi} + 1)} dx \right]_{l=1}. \quad (29)$$

The first term in the serie (8), is again equal to zero, as can be directly verified by computing the integral for g , deriving the result with respect to l and evaluating at $l = 1$. The second term needs

$$\frac{dg}{dx} = \frac{1}{\psi} \left[2(x - \psi)x^{3/2} + (x - \psi)^2 \frac{3}{2} x^{1/2} \right]. \quad (30)$$

Derivating this with respect to l and evaluating at $l = 1$ we obtain for the correction,

$$\frac{\pi^2}{3} \psi^{3/2}. \quad (31)$$

The third term needs the third derivative of g . This is found to be,

$$\frac{1}{\psi} \left[9x^{1/2} + x^{-1/2}(x - \psi) \frac{9}{2} - (x - \psi)^2 \frac{3}{8} x^{-3/2} \right]. \quad (32)$$

Again deriving with respect to l and evaluating at $l = 1$, it yields the following correction

$$\frac{7\pi^4}{360} 9\psi^{-1/2}. \quad (33)$$

Collecting the results, we get for $\langle U \rangle$,

$$\begin{aligned} \langle U \rangle &= \frac{5h^3}{8\pi V(2m)^{3/2} \mu^{5/2}} = \left(1 + \frac{5}{8} \left(\frac{\pi^2}{\beta\mu} \right)^2 - \frac{7}{384} \left(\frac{\pi}{\beta\mu} \right)^4 + \dots \right) \\ &\quad + \frac{q-1}{2} \left(0 + \frac{5\pi^2}{6} \frac{1}{\beta\mu} + \frac{315\pi^4}{720} \frac{1}{(\beta\mu)^3} + \dots \right). \end{aligned} \quad (34)$$

As above, this expression represents the correction due to non-extensivity that arise for the energy.

It is important to note that the $T \rightarrow 0$ limit is attained without any correction thus suggesting that non-extensivity can hardly have any role for very low temperatures. One can now study particular physical systems. For instance, one can immediately realize that the Chandrasekhar mass will be not modified at all in passing to a non-extensive context, at least up to order $(1 - q)$. This is simple because the Chandrasekhar mass is a construct that arises at $T = 0$ where both corrections, to $\langle N \rangle$ and to $\langle U \rangle$, are exactly zero. However, if non-extensive statistics is concerned in the study of stellar structures, corrections will unavoidably arise.

III. THE BOSON CASE

Now we analyze the boson generalized distribution function, namely,

$$n_{q[bosons]} = \frac{1}{e^{\beta(\epsilon - \mu)} - 1} + \frac{q-1}{2} \frac{[\beta(\epsilon - \mu)]^2 e^{\beta(\epsilon - \mu)}}{(e^{\beta(\epsilon - \mu)} - 1)^2}, \quad (35)$$

where again $\beta = 1/kT$, ϵ is the energy level, μ is the chemical potential. Then, using equation (35), one can obtain the average number of particles as

$$\frac{\langle N \rangle_q}{V} = \frac{2\pi(2m)^{3/2}}{h^3} \left[\int_0^\infty \frac{\epsilon^{1/2} d\epsilon}{e^{\beta(\epsilon - \mu)} - 1} + \frac{q-1}{2} \int_0^\infty \frac{\beta^2 (\epsilon - \mu)^2 e^{\beta(\epsilon - \mu)} \epsilon^{1/2} d\epsilon}{(e^{\beta(\epsilon - \mu)} - 1)^2} \right]. \quad (36)$$

As in the standard case, we have to separate the state with $\epsilon = 0$, which has zero weight in the integral (36). For this level of energy, the distribution function yields,

$$N_q(\epsilon = 0) = \frac{z}{1-z} \left[1 + \frac{q-1}{2} \frac{(\ln z)^2}{1-z} \right] \quad (37)$$

where $z = e^{\beta\mu}$ is the fugacity of the gas. In the case $z \ll 1$, the correction term goes to zero as $z(\ln z)^2$ does. If $z \rightarrow 1$, the correction goes to 1 as z , and results are neglectable in comparison with the first, diverging, term. This can be seen in Fig. 1.

Substitution of $x = \beta\epsilon$ and $\psi = \beta\mu$ in equation (36) yields,

$$\frac{(N_e)_q}{V} = \frac{2\pi(2mk)^{3/2}}{h^3} T^{3/2} \left[I_1 + \frac{q-1}{2} I_q \right] \quad (38)$$

where $(N_e)_q$ stands for the number of particles in the excited states ($\epsilon \neq 0$), and I_1 and I_q are defined to be

$$I_1 = \int_0^\infty \frac{x^{1/2} dx}{e^{x-\psi} - 1}, \quad I_q = \int_0^\infty \frac{(x-\psi)^2 e^{x-\psi} x^{1/2} dx}{(e^{x-\psi} - 1)^2}. \quad (39)$$

The integral I_1 is the one which appears in the standard, $q = 1$ case of the boson gas and therefore the solution of it is the known one [27],

$$I_1 = \Gamma(3/2)g_{3/2}(z), \quad (40)$$

where $g_n(z) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{z^{-1}e^x - 1} dx \simeq \sum_{s=1}^\infty (z^s/s^n)$ for small z and Γ is the usual Gamma function, $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx$. For $0 \leq z \leq 1$ and $\forall n, n > 1$, the functions $g_n(z)$ are bounded by the Riemann zeta functions, which yields for all z values of interest,

$$I_1 \leq \Gamma(3/2)\zeta(3/2). \quad (41)$$

On the other hand, the integral I_q is again somewhat cumbersome to solve, but after some algebra, similar to what we did in the previous section, we managed to find the analytical solution as

$$I_q = \Gamma(7/2)g_{5/2}(z) - 2\psi\Gamma(5/2)g_{3/2}(z) + \Gamma(3/2)\psi^2g_{1/2}(z). \quad (42)$$

To get the previous result one has to take into account the known relationship between the $g_n(z)$ and its derivatives,

$$g_{n-1}(z) = z \frac{\partial}{\partial z} g_n(z) = \frac{\partial}{\partial(\ln z)} g_n(z). \quad (43)$$

Recalling again the definitions ψ and z , we can write $\psi = \ln z$. Putting these solutions of the integrals in equation (38), one can obtain

$$\frac{(N_e)_q}{V} = \frac{2\pi(2mk)^{3/2}}{h^3} T^{3/2} \times \left\{ \Gamma(3/2)g_{3/2}(z) + \frac{q-1}{2} [\Gamma(7/2)g_{5/2}(z) - 2\Gamma(5/2)\ln z g_{3/2}(z) + \Gamma(3/2)(\ln z)^2 g_{1/2}(z)] \right\}, \quad (44)$$

which is the q -dependent solution for the number of particles in the excited states of boson systems, including the standard case as a special one if $q = 1$.

Like in the standard case, let us study each term of the correction in a separate fashion in order to find the bounded form of them. The first term is similar to I_1 and hence it is bounded by $\Gamma(7/2)\zeta(5/2)$. The second and the third terms are different from the first one in the sense that they do not take their largest values at $z = 1$, but instead, both of them tend to 0 when $z \rightarrow 0$ and $z \rightarrow 1$. This behavior can be seen in Fig. 2. The largest values are situated at $z_{max1} \simeq 0.447$ for the second term and $z_{max2} \simeq 0.175$ for the third.

Consequently, if all these bounds are put together, then the total number of particles in all excited states is also bounded by,

$$(N_e)_q \leq V \frac{2\pi(2mk)^{3/2}}{h^3} T^{3/2} \left\{ \Gamma(3/2)\zeta(3/2) + \frac{q-1}{2} \times [\Gamma(7/2)\zeta(5/2) - 2\Gamma(5/2)\ln(0.447)g_{3/2}(0.447) + \Gamma(3/2)(\ln(0.175))^2 g_{1/2}(0.175)] \right\} \quad (45)$$

which gives

$$(N_e)_q \leq V \frac{2\pi(2mk)^{3/2}}{h^3} T^{3/2} \{2.315 + (q-1)3.079\}. \quad (46)$$

If we now concentrate on Bose-Einstein condensation, then the condition for the appearance of it can be expressed as

$$N > (N_e)_q. \quad (47)$$

Alternatively, with constant N and V , this condition can be recasted in the form,

$$T < (T_c)_q = \frac{h^2}{(2\pi)^{2/3} 2mk} \left\{ \frac{N}{V [2.315 + (q-1)3.079]} \right\}^{2/3}, \quad (48)$$

or up to order 1 in $(q-1)$,

$$T < (T_c)_q = \frac{h^2}{(2\pi)^{2/3} 2mk} \left(\frac{N}{2.315V} \right)^{2/3} (1 + (q-1)0.886), \quad (49)$$

where $(T_c)_q$ is the q -dependent characteristic temperature of the Bose-Einstein condensation. It is easily seen that this result shows that the critical temperature decreases when $q < 1$, which is consistent with the previous result of Curilef [15]. Note that the standard $(T_c)_1$ case can easily be obtained for $q = 1$ value in the above expression.

Any accurate simultaneous determination of N , V and T can yield, in principle, a bound upon q . However in practice, this could be well below any practical possibility, due to the smallness of the correction term.

IV. CONCLUSIONS

Our main results can be summarized as follows. We have been able to solve, using the generalized quantal distribution functions in the factorization approach, the values for the average number of particles and energy in the case of system of non-interacting particles, either fermions or bosons. These results are expected to be useful in any analysis of statistical phenomena within the context of non-extensive scenarios. We could explicitly see that all the terms coming from the non-extensive part of the integrals go to zero when the temperature goes to zero, something that could be expected due to the form of n_q . Non-extensivity can not play a role for very low temperatures, at least up to order $(1-q)$. As a consequence, for instance, the well known Chandrasekhar limit for white dwarfs stars is not affected by a change of the statistical framework. However, it is to be explicitly stated that any other model of star, that happens with $T \neq 0$, will be affected by such change. This is why, for example, this statistical scenario could be useful to tackle the solar neutrino problem [28]. In the boson case, we have seen that a small correction appears to the Bose-Einstein condensation, this happening not exactly at $T = 0$. If this correction is enough to work out a possible bound upon q in this system remains to be studied. In the paper by Curilef it was argued that due to a possible fractility of the universe, distribution functions could be modified in the sense described here, and that this could be useful to study the behavior of diffuse gas clouds. This could be an example of where the $q \neq 1$ statistics may arise, although due to the form in which equilibrium distributions of boson stars arise it is unlikely that this could be applied for those cases.

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Figure Captions

Figure 1 : The plot of $N_q(\epsilon = 0)$ as a function of z for various q values.

Figure 2 : The plot of the second and third terms of the q -dependent part of eq.(43) as a function of z .

Figure 1

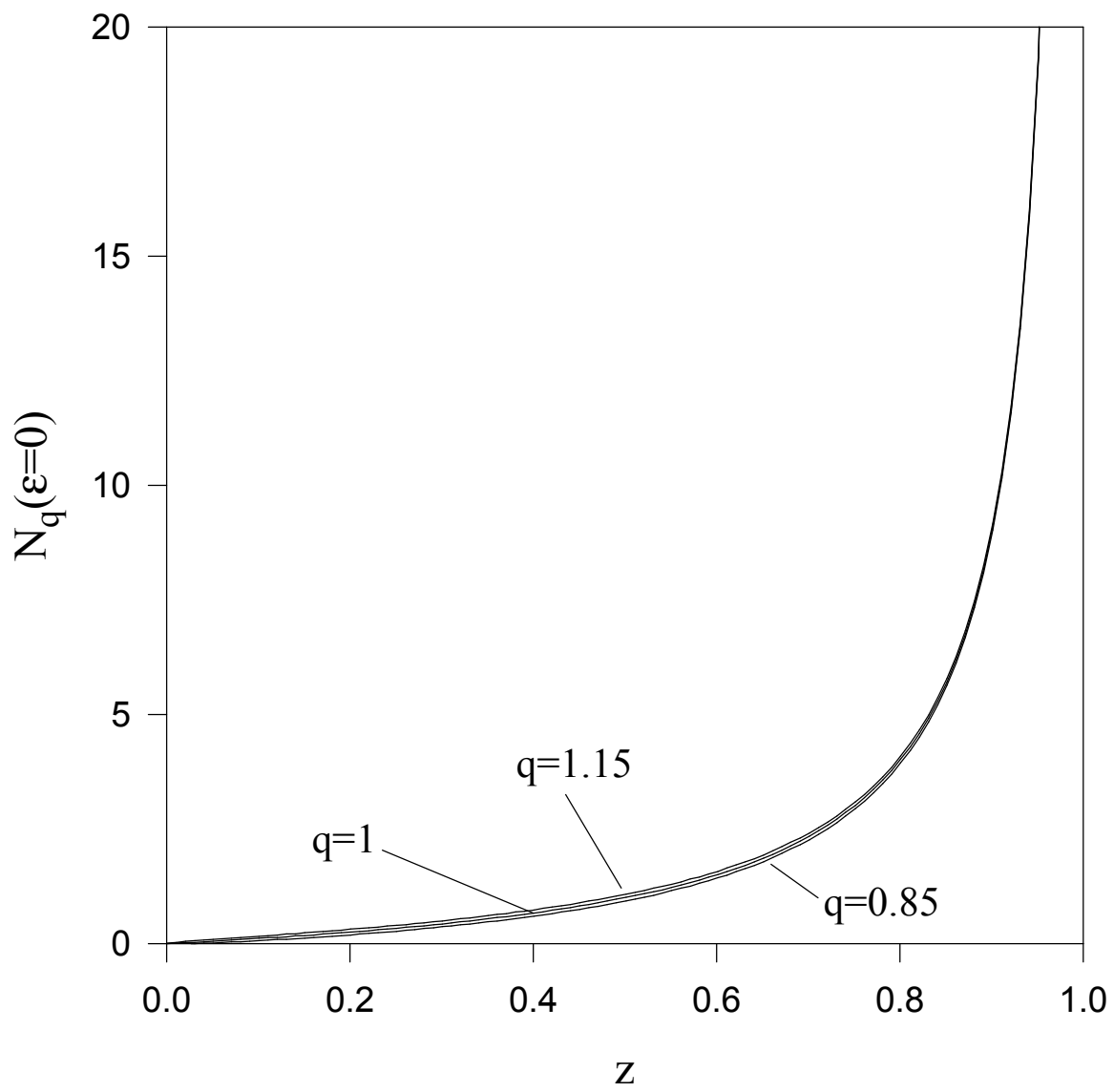


Figure 2

